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Polynomial Approximation in $E^p(D)$ with $0<p<1$

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In this paper, we construct approximants by means of interpolation polynomials to prove Jackson's theorem and the Bernstein inequality in $E^p(D)$ with $0<p<1$.

1. INTRODUCTION

Let D be a Jordan domain in the complex plane $\mathbb{C}$ with rectifiable boundary Γ. For $0<p<\infty$, we set

$$E^p(D)=\{f:(\psi')^{1/p}f\circ\psi\in H^p\},$$

where ψ is a Riemann mapping of the unit disk U onto D and H^p is the classical Hardy space for U [8].

For $f\in E^p(D)$, we define

$$\|f\|_{E^p(D)}=\|(\psi')^{1/p}f\circ\psi\|_{H^p}. \tag{1}$$

The problem of the degree of polynomial approximation in H^p for $1\leqslant p<\infty$ is not difficult, and Storoženko solved it in the case $0<p<1$ in 1970s [1, 2]. For spaces $E^p(D)$ in a Jordan domain, this problem has been studied by several authors when $1\leqslant p<\infty$ [3, 4]. The Faber operator was commonly used in these articles. In this paper, we shall study polynomial approximation in $E^p(D)$ when $0<p<1$, and approximants will be constructed directly by means of Lagrange interpolation polynomials. We shall use the modulus of continuity of $f\circ\psi$ to estimate the order of the approximation. However, Γ is required to be $3+\delta$ smooth, which means it has a $3+\delta$ smooth normal parametric representation.

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In this paper, c_j denote positive constants that only depend on p and D. The assumption that $0<p<1$ and that Γ is $3+\delta$ smooth is kept throughout the paper.

2. Some Preliminaries

A positive measure μ on U is called a Carleson measure if there exists a constant M such that

$$\mu(S(I)) \leqslant M\,|I| \tag{2}$$

for any interval $I \subset \partial U$, where

$$S(I)=\left\{re^{it} : 1-\frac{|I|}{2\pi} \leqslant r<1,\, e^{it} \in I\right\}.$$

It is well known that if μ is a Carleson measure, then for $g \in H^p$ we have [9]

$$\left\{\int_U |g|^p\, d\mu\right\}^{1/p} \leqslant 4(80)^4(M^2+1)\,\|g\|_{H^p}, \tag{3}$$

where M is the same constant as on the right of (2).

Besides the Riemann mapping $\psi : U \to D$, with the inverse $\phi : D \to U$, we also consider the Riemann mapping $\Psi: \mathbb{C}\backslash U \to \mathbb{C}\backslash D$ with $\Psi(\infty)=\infty$, $\Psi'(\infty)>0$, and let Φ be the inverse mapping of Ψ. Then ψ and Ψ (respectively, ϕ and Φ) can be extended to ∂U (respectively, Γ) $3+\delta$ smoothly. For $z, \zeta \in \mathbb{C}\backslash D$ and $u, v \in \mathbb{C}\backslash U$, we have

$$c_2^{-1} \leqslant \left|\frac{\psi(u)-\psi(v)}{u-v}\right| \leqslant c_2 \tag{4}$$

$$c_2^{-1} \leqslant \left|\frac{\Phi(z)-\Phi(\zeta)}{z-\zeta}\right| \leqslant c_2 \tag{5}$$

$$c_2^{-1} \leqslant |\psi'(u)| \leqslant c_2 \tag{6}$$

$$c_2^{-1} \leqslant |\Phi'(z)| \leqslant c_2. \tag{7}$$

By (1) we have

$$c_2^{-1/p}\,\|f\circ\psi\|_{H^p} \leqslant \|f\|_{E^p(D)} \leqslant c_2^{1/p}\,\|f\circ\psi\|_{H^p}.$$

We will not identify $\|f\|_{E^p(D)}$ and $\|f\circ\psi\|_{H^p}$, and $\|\cdot\|$ will denote either of these norms.

Let

$$D_n = \{\psi(rh_n(e^{i\theta})) : 0 \leqslant r < 1,\ -\pi \leqslant \theta < \pi\},$$

where

$$h_n(e^{i\theta}) = e^{i\theta} - \frac{\lambda(e^{i\theta})}{\sqrt{n}} \tag{8}$$

and

$$\lambda(e^{i\theta}) = \frac{e^{i\theta}}{|(\Phi \circ \psi)'(e^{i\theta})|}. \tag{9}$$

If n is sufficiently large, D_n is a Jordan domain bounded by the curve

$$\{\psi(h_n(e^{i\theta})) : -\pi \leqslant \theta < \pi\}.$$

Let $\Psi_n : \mathbb{C} \setminus U \to \mathbb{C} \setminus D_n$ be the Riemann mapping with $\Psi_n(\infty) = \infty$, $\Psi'_n(\infty) > 0$, and let Φ_n be the inverse mapping of Ψ_n.

LEMMA 1. *For* $z, \zeta \in \mathbb{C} \setminus D_n$

$$c_3^{-1} \leqslant \left| \frac{\Phi_n(z) - \Phi_n(\zeta)}{z - \zeta} \right| \leqslant c_3 \tag{10}$$

$$c_3^{-1} \leqslant |\Phi'_n(z)| \leqslant c_3 \tag{11}$$

and for $z \in \Gamma$

$$1 + \frac{1}{\sqrt{n}} - \frac{c_4}{n} \leqslant |\Phi_n(z)| \leqslant 1 + \frac{1}{\sqrt{n}} + \frac{c_4}{n}. \tag{12}$$

Proof. Let $z(\theta) = \psi(e^{i\theta})$ and $z_n(\theta) = \psi(h_n(e^{i\theta}))$ be the parametric representations of Γ and ∂D_n, respectively. It is not too difficult to verify

$$|z(\theta) - z_n(\theta)| \leqslant \frac{c_5}{\sqrt{n}}$$

$$c_5^{-1} \leqslant |z'(\theta)|, \qquad |z'_n(\theta)| \leqslant c_5$$

$$|z''(\theta)|, \qquad |z''_n(\theta)| \leqslant c_5$$

$$|z''(\theta) - z''_n(\theta)| \leqslant \frac{c_5}{\sqrt{n}}$$

$$|z''_n(\theta + t) - z''_n(\theta)| \leqslant c_5 t^{\delta}.$$

Note that the third derivatives of ψ and Φ appear in the term of $z_n''(\theta)$, but they are both bounded.

From a result due to Warschawski [5, Theorem 5], we have

$$|\Phi_n'(\psi(h_n(e^{i\theta}))) - \Phi'(\psi(e^{i\theta}))| \leqslant \frac{c_6}{\sqrt{n}}.$$

It follows that

$$|(\Phi_n \circ \psi)'(h_n(e^{i\theta})) - (\Phi \circ \psi)'(e^{i\theta})| \leqslant \frac{c_7}{\sqrt{n}}. \tag{13}$$

By Warschawski's other conclusion [6, Theorem 5], we have (10), (11), and

$$|\Psi_n''(\zeta)| \leqslant c_8, \qquad \zeta \in \mathbb{C} \backslash D_n. \tag{14}$$

For $z = \psi(e^{i\theta}) \in \Gamma$, let us denote by σ the segment from $h_n(e^{i\theta})$ to $e^{i\theta}$. Then by (8) we have

$$\Phi_n(z) = \Phi_n \circ \psi(h_n(e^{i\theta})) + (\Phi_n \circ \psi)'(h_n(e^{i\theta})) \frac{\lambda(e^{i\theta})}{\sqrt{n}}$$
$$+ \int_\sigma [(\Phi_n \circ \psi)'(u) - (\Phi_n \circ \psi)'(h_n(e^{i\theta}))] \, du.$$

Since the length of σ equals $|\lambda(e^{i\theta})|/\sqrt{n}$, and by (14) we have

$$\Phi_n(z) = \Phi_n \circ \psi(h_n(e^{i\theta})) + \frac{e^{i\theta}(\Phi_n \circ \psi)'(h_n(e^{i\theta}))}{\sqrt{n}\, |(\Phi \circ \psi)'(e^{i\theta})|} + O\left(\frac{1}{n}\right).$$

By (13)

$$\Phi_n(z) = \Phi_n \circ \psi(h_n(e^{i\theta})) + \frac{e^{i\theta}(\Phi \circ \psi)'(e^{i\theta})}{\sqrt{n}\, |(\Phi \circ \psi)'(e^{i\theta})|} + O\left(\frac{1}{n}\right). \tag{15}$$

Since $\Phi_n \circ \psi(h_n(e^{i\theta}))$ is on the unit circle, we assume

$$e^{it} = \Phi_n \circ \psi(h_n(e^{i\theta}));$$

taking the derivative with respect to t, we have

$$ie^{it} = (\Phi_n \circ \psi)'(h_n(e^{i\theta})) \frac{dh_n(e^{i\theta})}{d\theta} \frac{d\theta}{dt}.$$

It follows that

$$\frac{\pi}{2}+t=\arg(\Phi_n\circ\psi)'(h_n(e^{i\theta}))+\arg\frac{dh_n(e^{i\theta})}{d\theta}+\arg\frac{d\theta}{dt}. \tag{16}$$

Since $d\theta/dt>0$, then $\arg(d\theta/dt)=0$. Obviously

$$\begin{aligned}\arg\frac{dh_n(e^{i\theta})}{d\theta}&=\arg\left(ie^{i\theta}+\frac{1}{\sqrt{n}}\frac{\lambda(e^{i\theta})}{d\theta}\right)\\&=\frac{\pi}{2}+\theta+O\left(\frac{1}{\sqrt{n}}\right)\end{aligned}$$

and by (13)

$$\arg(\Phi\circ\psi)'(h_n(e^{i\theta}))=\arg(\Phi_n\circ\psi)'(e^{i\theta})+O\left(\frac{1}{\sqrt{n}}\right).$$

Together with (16), we have

$$t=\theta+\arg(\Phi\circ\psi)'(e^{i\theta})+O\left(\frac{1}{\sqrt{n}}\right).$$

It follows that

$$\frac{e^{i\theta}(\Phi\circ\psi)'(e^{i\theta})}{|(\Phi\circ\psi)'(e^{i\theta})|}=e^{it}+O\left(\frac{1}{\sqrt{n}}\right).$$

By (15),

$$\Phi_n\circ\psi(e^{i\theta})=e^{it}+\frac{1}{\sqrt{n}}e^{it}+O\left(\frac{1}{n}\right).$$

This follows (12) and completes the proof of Lemma 1. ∎

Set

$$r_n=1+\frac{1}{\sqrt{n}}-\frac{2c_4}{n} \tag{17}$$

and set

$$\rho_n=1+\frac{1}{\sqrt{n}}+\frac{2c_4}{n}. \tag{18}$$

For n sufficiently large such that $r_n>1$, we denote

$$\gamma_n=\{\Psi_n(r_n e^{i\theta}):-\pi\leqslant\theta<\pi\}$$

and

$$\Gamma_n = \{ \Psi_n(\rho_n e^{i\theta}) : -\pi \leqslant \theta < \pi \}.$$

From Lemma 1, we have

$$\frac{c_6^{-1}}{n} \leqslant \inf_{\substack{z \in \Gamma \\ \zeta \in \gamma_n \cup \Gamma_n}} |z - \zeta| \leqslant \frac{c_6}{n}. \tag{19}$$

Then for $\zeta \in \gamma_n$, we have

$$1 - \frac{c_7}{n} \leqslant |\phi(\zeta)| \leqslant 1 - \frac{c_7^{-1}}{n}. \tag{20}$$

Let G_n be the domain enclosed by γ_n and let K_n be the domain bounded by γ_n and Γ_n; that means

$$K_n = \{ z : r_n < |\Phi_n(z)| < \rho_n \}.$$

For n sufficiently large, it is obvious

$$D_n \subset G_n \subset D \subset G_n \cup \bar{K}_n.$$

LEMMA 2. *For $F \in E^p(D)$, we have*

$$\int_{\gamma_n} |F(z)| \, |dz| \leqslant c_8 n^{1/p - 1} \, \|F\|_p. \tag{21}$$

Proof. It is known [1]

$$\max_{|u| = r} |g(u)| \leqslant (1 - r)^{-1/p} \, \|g\|_p$$

holds for $g \in H^p$, $0 < r < 1$. By (20)

$$\max_{z \in \gamma_n} |F(z)| \leqslant c_7 n^{1/p} \, \|F\|_p.$$

Since $\phi(\gamma_n)$ is the image of the circle $|u| = r_n$ under the smooth mapping $\phi \circ \Psi_n$, the arc measure on it is a Carleson measure. By (3)

$$\int_{\phi(\gamma_n)} |F \circ \psi(u)|^p \, |du| \leqslant c_9 \, \|F\|_p^p.$$

Hence

$$\int_{\gamma_n} |F(z)| \, |dz| \leqslant \max_{z \in \gamma_n} |F(z)|^{1-p} \int_{\phi(\gamma_n)} |F \circ \psi(u)|^p \, |\psi(u)| \, |du|$$
$$\leqslant c_8 n^{1/p - 1} \, \|F\|_p. \quad \blacksquare$$

3. Construction of Approximants

Let Π_n be the set of polynomials of degree at most n. For $f \in E^p(D)$, we define

$$E_n(f)_p = \inf_{P_n \in \Pi_n} \|f - P_n\|_p.$$

Set

$$u_{k,j}^{(n)} = \left(1 + \frac{1}{2\sqrt{n}}\right) \exp\left(\frac{2\pi j}{k+1} i\right), \qquad j = 0, 1, ..., k.$$

They are the roots of

$$u^{k+1} - \left(1 + \frac{1}{2\sqrt{n}}\right)^{k+1} = 0.$$

Let

$$z_{k,j}^{(n)} = \Psi_n(u_{h,j}^{(n)});$$

then $z_{k,j}^{(n)} \in G_n \subset D$.

For $f \in E^p(D)$, we denote by $L_{n,k}(f, z)$ the kth Lagrange interpolation polynomial to f at the points $\{z_{k,j}^{(n)}, j = 0, 1, ..., k\}$. That means $L_{n,k}(f, z) \in \Pi_k$ and

$$L_{n,k}(f, z_{k,j}^{(n)}) = z_{k,j}^{(n)}, \qquad j = 1, 2, ..., k.$$

Let

$$\omega_{n,k}(z) = \prod_{j=1}^{k} (z - z_{k,j}^{(n)}).$$

Then

$$L_{n,k}(f, z) = \frac{1}{2\pi i} \int_{\gamma_n} \frac{\omega_{n,k}(\zeta) - \omega_{n,k}(z)}{\omega_{n,k}(\zeta)} \frac{f(\zeta)}{\zeta - z} d\zeta. \tag{22}$$

Choosing $l = 1 + [2/p]$, we define $\{A_n^k\}$ by the identity

$$\left(\frac{1 - x^{n+1}}{1 - x}\right)^l = \sum_{k=0}^{ln} A_{n,k} x^k. \tag{23}$$

Of course, $A_{n,k}$ are all positive integers. Taking $x \to 1$, we can see

$$\sum_{k=0}^{ln} A_{n,k} = (n+1)^l.$$

Set

$$V_n(f, z) = \frac{1}{(n+1)} \sum_{k=n}^{(l+1)n} A_{n,k-n} L_{n,k}(f, z). \tag{24}$$

Obviously $V_n(f, z) \in \Pi_{(l+1)n}$.

For $g \in H^p$, the modulus of continuity in H^p is defined by

$$\omega(g, t)_p = \sup_{0<s<t} \| g(ue^{is}) - g(u) \|_{H^p}.$$

Now we state our results.

THEOREM 1. *Suppose* $0 < p < 1$ *and* Γ *is* $3+\delta$ *smooth. Then for* $f \in E^p(D)$,

$$\|f(z) - V_n(f, z)\|_p \leqslant c_{10} \omega\left(f \circ \psi, \frac{1}{n}\right)_p \tag{25}$$

and it follows that

$$E_n(f)_p \leqslant c_{11} \omega\left(f \circ \psi, \frac{1}{n}\right)_p. \tag{26}$$

From Theorem 1, we can obtain the so-called "de la Vallee Poussin theorem" in $E^p(D)$.

COROLLARY 1. *Under the conditions of Theorem* 1,

$$\|f(z) - V_n(f, z)\|_p \leqslant c_{12} E_n(f)_p.$$

The reason is $V_n(P_n, z) = P_n(z)$ for $P_n(z) \in \Pi_n$ and

$$\|f(z) - V_n(f, z)\|_p \leqslant 2c_{10} \|f\|_p.$$

It is known that (25) is sharp in the case $D = U$ [1]. However, we will give the Bernstein inequality in $E^p(D)$, which means that (25) is even sharp in the general cases.

THEOREM 2. *Under the conditions of Theorem* 1,

$$\|P_n'\|_p \leqslant c_{13} n \|P_n\|_p$$

holds for any $P_n \in \Pi_n$.

As in [2], Theorem 2 implies the inverse theorem of approximation in $E^p(D)$.

COROLLARY 2. *Under the conditions of Theorem* 1,

$$\omega\left(f\circ\psi, \frac{1}{n}\right)_p \leqslant \frac{c_{13}}{n}\left\{\sum_{j=1}^{n} j^{p-1}E_j(f)_p^p\right\}^{1/p}.$$

For the sake of simplicity, we shall use the notations $L_k(f, z)$, $\omega_k(z)$, $u_{k,j}$, and A_k to denote $L_{n,k}(f, z)$, $u_{k,j}^{(n)}$, and $A_{n,k}$, respectively. Before proving the theorems, we need to prove the asymptotic behaviour of $\omega_k(z)$.

4. THE ASYMPTOTIC BEHAVIOUR

LEMMA 3. *Let* $n \leqslant k \leqslant (l+1)n$. *Then for* $z \in \mathbb{C}\setminus D_n$,

$$\left|\frac{\omega_k(z)}{d_n^{k+1}\{[\Phi_n(z)]^{k+1} - (1+(1/2\sqrt{n}))^{k+1}\}} - 1\right| \leqslant c_{15}e^{-\sqrt{n}/4},$$

where $d_n = \Psi_n(\infty)$.

Proof. As in [7], for $u, v \in \mathbb{C}\setminus D_n$, we construct

$$\chi_n(u, w) = \begin{cases} \dfrac{\Psi_n(u) - \Psi_n(v)}{d_n(u-v)}, & v \neq u \\ \dfrac{\Psi_n'(u)}{d_n}, & v = u. \end{cases}$$

Let $\log \chi_n(u, v)$ denote the branch of logarithm for which $\log \chi_n(u, \infty) = 0$. By (10) we have

$$|\log \chi_n(u, v)| \leqslant c_{16}$$

and we have the Laurrent series

$$\log \chi_n(u, v) = \sum_{m=1}^{\infty} \frac{a_{n,m}(u)}{v^m}.$$

Evidently

$$\begin{aligned} |a_{n,m}(u)| &= \left|\frac{1}{2\pi i}\int_{|v|=1} v^{m-1}\log \chi_n(u, v)\, dv\right| \\ &\leqslant c_{16}. \end{aligned} \tag{27}$$

For $z=\Psi_n(u)$, we have

$$\log \frac{\omega_k(z)}{d_n^{k+1}[u^{k+1}-(1+(1/2\sqrt{n}))^{k+1}]}$$
$$=\sum_{j=0}^{k} \log \chi_n(u, u_{k,j})$$
$$=\sum_{m=1}^{\infty} a_{n,m}(u) \sum_{j=0}^{k} (u_{k,j})^{-m}$$
$$=(k+1) \sum_{N=1}^{\infty} a_{n,(k+1)N}(u) \left(1-\frac{1}{2\sqrt{n}}\right)^{-(k+1)N}$$

and by (27) we have

$$\left|\log \frac{\omega_k(z)}{d_n^{k+1}\{[\Phi_n(z)]^{k+1}-(1+(1/2\sqrt{n}))^{k+1}\}}\right|$$
$$\leqslant c_{16}(k+1) \sum_{N=1}^{\infty} \left(1-\frac{1}{2\sqrt{n}}\right)^{-nN}$$
$$\leqslant c_{15}\, e^{-\sqrt{n}/4}.$$

This completes the proof of Lemma 3. ▮

Set

$$H_k(\zeta, z)=\frac{\omega_k(z)}{\omega_k(\zeta)}-\left[\frac{\Phi_n(z)}{\Phi_n(\zeta)}\right]^{k+1}.$$

LEMMA 4. *Let $n \leqslant k \leqslant (l+1)n$. Then for $\zeta, z \in \bar{K}_n$*

$$\left|\frac{\Phi_n(z)}{\Phi_n(\zeta)}\right|^{k+1} \leqslant c_{17}$$
$$|H_k(\zeta, z)| \leqslant c_{17}\, e^{-\sqrt{n}/4} \tag{28}$$

and for $z \in \Gamma$, $\zeta \in \bar{K}_n$, we also have

$$\left|\frac{\partial H_k(\zeta, z)}{\partial z}\right| \leqslant c_{17}\, e^{-\sqrt{n}/5}. \tag{29}$$

Proof. For $\zeta, z \in \bar{K}_n$, we have

$$r_n \leqslant |\Phi_n(\zeta)|, \qquad |\Phi_n(z)| \leqslant \rho_n.$$

It follows that

$$\left|\frac{\Phi_n(z)}{\Phi_n(\zeta)}\right|^{k+1} \leqslant \left(\frac{\rho_n}{r_n}\right)^{(l+1)n} \leqslant c_{17}.$$

From Lemma 3

$$\left|\frac{\omega_n(z)}{d_n^{k+1}} - [\Phi_n(z)]^{k+1}\right| \leqslant c_{18}\, e^{-\sqrt{n}/4}\, |\Phi_n(z)|^{k+1} + \left(1 + \frac{1}{2\sqrt{n}}\right)^{k+1}$$
$$\leqslant c_{19}\, e^{-\sqrt{n}/4}\, |\Phi_n(z)|^{k+1}.$$

Then we also have

$$\left|\frac{\omega_n(\zeta)}{d_n^{k+1}} - [\Phi_n(\zeta)]^{k+1}\right| \leqslant c_{19}\, e^{-\sqrt{n}/4}\, |\Phi_n(\zeta)|^{k+1}.$$

For n sufficiently large, we have

$$|\Phi_n(\zeta)|^{k+1} \leqslant 2\left|\frac{\omega_n(\zeta)}{d_n^{k+1}}\right|.$$

Then

$$\left|\frac{d_n^{k+1}}{\omega_n(\zeta)} - [\Phi_n(\zeta)]^{-(k+1)}\right| \leqslant 2c_{19}\, e^{-\sqrt{n}/4}\, |\Phi_n(\zeta)|^{-(k+1)}.$$

Hence

$$\left|\frac{\omega_k(z)}{\omega_k(\zeta)} - \left[\frac{\Phi_n(z)}{\Phi_n(\zeta)}\right]^{k+1}\right|$$
$$\leqslant \left|\frac{\omega_k(z)}{\omega_k(\zeta)} - \frac{d_n^{k+1}[\Phi_n(z)]^{k+1}}{\omega_n(\zeta)}\right|$$
$$+ |\Phi_n(z)|^{k+1}\left|\frac{d_n^{k+1}}{\omega_n(\zeta)} - [\Phi_n(\zeta)]^{-(k+1)}\right|$$
$$\leqslant c_{17}\, e^{-\sqrt{n}/4}.$$

Then we have (28).

Since $H_n(\zeta, z)$ is analytic with respect to z in $\bar{K}_n$, we have

$$\frac{\partial H_k(\zeta, z)}{\partial z} = \frac{1}{2\pi i}\int_{\Gamma_n \cup \gamma_n} \frac{H_k(\zeta, \tau)}{(\tau - z)^2}\, d\tau.$$

By (19) and (28), for $z \in \Gamma$ we have

$$\left|\frac{\partial H_k(\zeta, z)}{\partial z}\right| \leqslant c_{20} n^2 e^{-\sqrt{n}/4}.$$

This implies (29). ∎

5. Proof of the Theorems

Proof of Theorem 1. For $z \in \Gamma$, located in the exterior of γ_n, we have

$$\frac{1}{2\pi i}\int_{\gamma_n} \frac{f(\zeta)}{\zeta - z}\, d\zeta = 0.$$

By (22)

$$L_k(f, z) = -\frac{1}{2\pi i}\int_{\gamma_n} \frac{\omega_k(z)}{\omega_k(\zeta)} \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

It follows that

$$V_n(f, z) = -\frac{1}{(n+1)^l} \sum_{k=n}^{(l+1)n} A_{k-n} \frac{1}{2\pi i}\int_{\gamma_n} \frac{\omega_k(z)}{\omega_k(\zeta)} \frac{f(\zeta)}{\zeta - z}\, d\zeta. \tag{30}$$

Evidently

$$\begin{aligned} f(z) - V_n(f, z) &= \frac{1}{(n+1)^l} \sum_{k=n}^{(l+1)n} A_{k-n} \frac{1}{2\pi i}\int_{\gamma_n} \frac{\omega_k(z)}{\omega_k(\zeta)} \frac{f(\zeta) - f(z)}{\zeta - z}\, d\zeta \\ &= \frac{1}{(n+1)^l} \sum_{k=n}^{(l+1)n} A_{k-n} \frac{1}{2\pi i}\int_{\gamma_n} H_k(\zeta, z) \frac{f(\zeta) - f(z)}{\zeta - z}\, d\zeta \\ &\quad + \frac{1}{(n+1)^l} \sum_{k=n}^{(l+1)n} A_{k-n} \frac{1}{2\pi i}\int_{\gamma_n} \left[\frac{\Phi_n(z)}{\Phi_n(\zeta)}\right]^{k+1} \frac{f(\zeta) - f(z)}{\zeta - z}\, d\zeta \\ &= I_1(z) + I_2(z). \end{aligned} \tag{31}$$

By (28)

$$\begin{aligned} |I_1(z)| &\leqslant \frac{1}{(n+1)^l} \sum_{k=n}^{(l+1)n} A_{k-n} \frac{1}{2\pi}\int_{\gamma_n} |H_k(\zeta, z)| \left|\frac{f(\zeta) - f(z)}{\zeta - z}\right| |d\zeta| \\ &\leqslant c_{17} e^{-\sqrt{n}/4} \int_{\gamma_n} \left|\frac{f(\zeta) - f(z)}{\zeta - z}\right| |d\zeta| \\ &\leqslant \frac{c_{20}}{(n+1)^l} \int_{\gamma_n} \frac{|f(\zeta) - f(z)|}{|\zeta - z|^{l+1}} |d\zeta|, \qquad z \in \Gamma. \end{aligned} \tag{32}$$

By (23), note $x = \Phi_n(z)/\Phi_n(\zeta)$

$$|I_2(z)| = \frac{1}{(n+1)^l} \left| \frac{1}{2\pi i} \int_{\gamma_n} \left(\frac{1 - x^{n+1}}{1 - x} \right)^l x^{n+1} \frac{f(\zeta) - f(z)}{\zeta - z} \, d\zeta \right|$$
$$\leqslant \frac{1}{2\pi(n+1)^l} \int_{\gamma_n} \frac{(1 + |x|^{n+1})^l}{|1 - x|^l} |x|^{n+1} \left| \frac{f(\zeta) - f(z)}{\zeta - z} \right| |d\zeta|$$
$$\leqslant \frac{c_{21}}{(n+1)^l} \int_{\gamma_n} \frac{|f(\zeta) - f(z)|}{|\zeta - z|^{l+1}} |d\zeta|.$$

By (10) we have

$$|\zeta - z| \leqslant c_3 \, |1 - x|$$

and from Lemma 3 we have

$$|I_2(z)| \leqslant \frac{c_{22}}{(n+1)^l} \int_{\gamma_n} \frac{|f(\zeta) - f(z)|}{|\zeta - z|} |d\zeta|. \tag{33}$$

Thus

$$|f(z) - V_n(f, z)| \leqslant |I_1(z)| + |I_2(z)|$$
$$\leqslant \frac{c_{20} + c_{22}}{(n+1)^l} \int_{\gamma_n} \frac{|f(\zeta) - f(z)|}{|\zeta - z|^{l+1}} |d\zeta|, \qquad z \in \Gamma. \tag{34}$$

By (20), for $z \in \Gamma$ and $\zeta \in \gamma_n$, we have

$$|[\phi(\zeta)]^n - [\phi(z)]^n| \geqslant |\phi(z)|^n - |\phi(\zeta)|^n$$
$$\geqslant 1 - \left(1 - \frac{c_7^{-1}}{n}\right)^n$$
$$\geqslant c_{23}.$$

Therefore by (34) we have

$$|f(z) - V_n(f, z)| \leqslant \frac{c_{24}}{n+1} \int_{\gamma_n} \left| \frac{[\phi(\zeta)]^n - [\phi(z)]^n}{\phi(\zeta) - \phi(z)} \right|^{l+1} |f(\zeta) - f(z)| \, |d\zeta|.$$

Let

$$F_z(\zeta) = \left\{ \frac{[\phi(\zeta)]^n - [\phi(z)]^n}{\phi(\zeta) - \phi(z)} \right\}^{l+1} [f(\zeta) - f(z)].$$

From Lemma 2

$$|f(z)-V_n(f,z)| \leqslant \frac{c_{24}}{n^{l+1-1/p}} \|F_z\|_p.$$

Then

$$\begin{aligned}
&\|f(z)-V_n(f,z)\|_p^p \\
&\leqslant \frac{c_{25}}{n^{(l+1)p-1}} \int_\Gamma |dz| \int_\Gamma \left|\frac{[\phi(\zeta)]^n-[\phi(z)]^n}{\phi(\zeta)-\phi(z)}\right|^{(l+1)p} |f(\zeta)-f(z)|^p\,|d\zeta| \\
&\leqslant \frac{c_{26}}{n^{(l+1)p-1}} \int_{-\pi}^{\pi} d\theta \int_{-\pi}^{\pi} \left|\frac{e^{int}-e^{in\theta}}{e^{it}-e^{i\theta}}\right|^{(l+1)p} |f\circ\psi(e^{it})-f\circ\psi(e^{i\theta})|^p\,dt.
\end{aligned}$$

As in [2], this follows (32) and completes the proof Theorem 1. ∎

Proof of Theorem 2. Since $V_n(P_n, z)=P_n(z)$ for $P_n\in\Pi_n$. By (30) we have

$$\begin{aligned}
P_n(z) &= -\frac{1}{(n+1)^l}\sum_{k=n}^{(l+1)n} A_{k-n}\frac{1}{2\pi i}\int_{\gamma_n}\frac{\omega_k(z)}{\omega_k(\zeta)}\frac{P_n(\zeta)}{\zeta-z}\,d\zeta \\
&= -\frac{1}{(n+1)^l}\sum_{k=n}^{(l+1)n} A_{k-n}\frac{1}{2\pi i}\int_{\gamma_n} H_k(\zeta,z)\frac{P_n(\zeta)}{\zeta-z}\,d\zeta \\
&\quad -\frac{1}{2\pi i(n+1)^l}\int_{\gamma_n}\left(\frac{1-x^{n+1}}{1-x}\right)^l x^n\frac{P_n(\zeta)}{\zeta-z}\,d\zeta,
\end{aligned}$$

where $x=\Phi_n(z)/\Phi_n(\zeta)$,

$$\begin{aligned}
P_n'(z) &= -\frac{1}{(n+1)^l}\sum_{k=n}^{(l+1)n} A_{k-n}\frac{1}{2\pi i}\int_{\gamma_n}\frac{\partial H_k(\zeta,z)}{\partial z}\frac{P_n(\zeta)}{\zeta-z}\,d\zeta \\
&\quad -\frac{1}{(n+1)^l}\sum_{k=n}^{(l+1)n} A_{k-n}\frac{1}{2\pi i}\int_{\gamma_n} H_k(\zeta,z)\frac{P_n(\zeta)}{(\zeta-z)^2}\,d\zeta \\
&\quad -\frac{1}{2\pi i(n+1)^l}\int_{\gamma_n}\frac{\partial}{\partial z}\left(\frac{1-x^{n+1}}{1-x}\right)^l x^n\frac{P_n(\zeta)}{\zeta-z}\,d\zeta \\
&\quad -\frac{1}{2\pi i(n+1)^l}\int_{\gamma_n}\left(\frac{1-x^{n+1}}{1-x}\right)^l x^n\frac{P_n(\zeta)}{(\zeta-z)^2}\,d\zeta \\
&= J_1(z)+J_2(z)+J_3(z)+J_4(z).
\end{aligned}$$

For $z\in\Gamma$, from Lemma 4 we have

$$|J_1(z)|+|J_2(z)| \leqslant \frac{c_{25}}{(n+1)^{l-1}}\int_{\gamma_n}\frac{|P_n(\zeta)|}{|\zeta-z|^{l+1}}\,|d\zeta|.$$

Similar to estimating $|I_2(z)|$ in the proof of Theorem 1

$$|J_4(z)| \leqslant \frac{c_{26}}{(n+1)^l} \int_{\gamma_n} \frac{|P_n(\zeta)|}{|\zeta - z|^{l+2}} |d\zeta| \leqslant \frac{c^{27}}{(n+1)^{l-1}} \int_{\gamma_n} \frac{|P_n(\zeta)|}{|\zeta - z|^{l+1}} |d\zeta|.$$

Evidently

$$|J_3(z)| = \frac{1}{2\pi(n+1)^l} \left| \int_{\gamma_n} \left[-n - 1 + (n+1)(l+1)\, x^{n+1} - \frac{l(1-x^{n+1})x}{1-x} \right] \times \frac{(1-x^{n+1})^{l-1}}{(1-x)^l} x^n \frac{\Phi_n'(z)}{\Phi_n(\zeta)} \frac{P_n(\zeta)}{\zeta - z} d\zeta \right|$$
$$\leqslant \frac{c_{28}}{(n+1)^l} \int_{\gamma_n} \left[(n+1) + \frac{1}{|\zeta - z|} \right] \frac{|P_n(\zeta)|}{|\zeta - z|^{l+1}} |d\zeta| \leqslant \frac{c_{29}}{(n+1)^{l-1}} \int_{\gamma_n} \frac{|P_n(\zeta)|}{|\zeta - z|^{l+1}} |d\zeta|.$$

Then we have

$$|P_n'(z)| \leqslant \frac{1}{(n+1)^l} \sum_{j=1}^{4} |J_j(z)| \leqslant \frac{c_{30}}{(n+1)^{l-1}} \int_{\gamma_n} \frac{|P_n(\zeta)|}{|\zeta - z|^{l+1}} |d\zeta|, \qquad z \in \Gamma.$$

Comparing with (34) in the proof of Theorem 1, we can obtain (26) with a similar procedure. ∎

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